§4 Series

I) Review

Let $\{z_n\}$ be a sequence of complex numbers.

Define $s_n = \sum_{k=1}^{n} z_k$ (partial sum)

If $\lim_{n\to\infty} s_n = S$, then we say that the infinite series $z_1+z_2+\cdots+z_n+\cdots$ converges to S and we write $\sum_{k=1}^{\infty} z_k = S$.

FACT :

Let $\rho_n = S - s_n$, then $\lim_{n\to\infty} s_n = S$ if and only if $\lim_{n\to\infty} \rho_n = 0$.

Further if $z_n = x_n + iy_n$, $n = 1, 2, \cdots$, we have

Theorem : $\sum_{k=1}^{\infty} z_k = S$ if and only if $\sum_{k=1}^{\infty} x_k = X$ and $\sum_{k=1}^{\infty} y_k = Y$,

where $S = X + iY$.

Comparsion Test

If $\{a_n\}, \{b_n\}$ are sequences of real numbers such that

1) $a_n \geq b_n \geq 0$ for all $n \in \mathbb{N}$ (or $n \geq N$)

2) $\sum_{n=1}^{\infty} a_n$ exists

then $\sum_{n=1}^{\infty} b_n$ exists and $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$

Theorem :

Let $\{z_n\}$ be a sequence of complex numbers.

If $\sum_{k=1}^{\infty} |z_k|$ converges (called absolutely convergent series), then $\sum_{k=1}^{\infty} z_k$ converges

proof : $\sum_{k=1}^{\infty} |z_k|$ converges

$\Leftrightarrow \sum_{k=1}^{\infty} \sqrt{x_k^2 + y_k^2}$ converges

$\sqrt{x_k^2+y_k^2} \geq |x_k|, |y_k|$ + Compasion test

$\Rightarrow \sum_{k=1}^{\infty} |x_k|$ and $\sum_{k=1}^{\infty} |y_k|$ converge

$\Rightarrow \sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ converge (Result in real case !)

$\Rightarrow \sum_{k=1}^{\infty} z_k$ converges.

But the converse of the theorem is NOT true !

If $z_n = (-1)^{n+1}\frac{1}{n}$, then

$$\begin{aligned} z_1 + z_2 &= 1 - \frac{1}{2} & &\le \frac{1}{1^2} \\ z_3 + z_4 &= \frac{1}{3} - \frac{1}{4} & &\le \frac{1}{3^2} \\ &\vdots \\ z_{2n-1} + z_{2n} &= \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)} & &\le \frac{1}{(2n-1)^2} \\ &\vdots \end{aligned}$$

$\therefore \sum_{k=1}^{\infty} z_k$ converges (by comparsion test)

But $\sum_{k=1}^{\infty} |z_k| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges. (In fact, example in real case)

Power Series

Series of the form (depending on z) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $z_0, a_n \in \mathbb{C}$,

is called a power series.

II) Talyor Series

Theorem :

Suppose that a function f is analytic throughout an open disk $\{z \in \mathbb{C} : |z-z_0| < R\}$.

Then, at each point z in that disk, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Rewrite :

$\forall z \in \{z \in \mathbb{C} : |z-z_0| < R\}$, $\varepsilon > 0$, $\exists N(z,\varepsilon) \in \mathbb{N}$ s.t. $\left|\sum_{k=1}^{n} a_k (z-z_0)^k - f(z)\right| < \varepsilon \quad \forall n \geq N(z,\varepsilon)$.

Approximate $f(z)$ by $\sum_{n=0}^{N} a_n (z-z_0)^n$, then the error ε can be arbitrary small by choosing sufficiently large N (depending on both z and ε). (pointwise convergent)

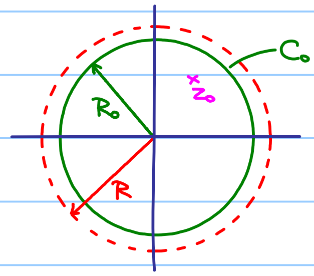
proof :

Firstly, prove the case $z_0 = 0$.

Fix $z \in \{z \in \mathbb{C} : |z| < R\}$.

Draw a circle C_0 such that

$$|z| < R_0 < R$$



Claim : $\rho_n(z) = f(z) - \sum_{k=0}^{n-1} a_k z^k$ tends to 0 as n tends to ∞.

By Cauchy Integral Formula,

$$\sum_{k=0}^{n-1} a_k z^k = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{n-1} \frac{z^k}{2\pi i} \int_{C_0} \frac{f(s)}{s^{k+1}} ds$$

Note : $f^{(k)}(0) = \frac{k!}{2\pi i} \int_{C_0} \frac{f(s)}{s^{k+1}} ds$

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{z-s} ds$$

$$\therefore \ f(z) - \sum_{k=0}^{n} a_k z^k = \frac{1}{2\pi i} \int_{C_0} f(s) \left[\frac{1}{s-z} - \frac{1}{s}\sum_{k=0}^{n-1} \left(\frac{z}{s}\right)^k\right] ds$$

$$\frac{1}{s-z} - \frac{1}{s}\sum_{k=0}^{n-1}\left(\frac{z}{s}\right)^k = \frac{1}{s-z} - \frac{1}{s}\,\frac{1-\left(\frac{z}{s}\right)^n}{1-\frac{z}{s}} = \frac{1}{s-z}\left(\frac{z}{s}\right)^n$$

$$\left|f(z) - \sum_{k=0}^{n} a_k z^k\right| \leq \frac{1}{2\pi} (2\pi R_0) \underbrace{\left[\frac{1}{R_0 - |z|} \left(\frac{|z|}{R_0}\right)^n\right]}_{} M$$

$\frac{|z|}{R_0} < 1 \Rightarrow \left(\frac{|z|}{R_0}\right)^n \to 0$ as $n \to \infty$

$$\left|\frac{1}{s-z} - \frac{1}{s}\sum_{k=0}^{n-1}\left(\frac{z}{s}\right)^k\right| = \frac{1}{|s-z|}\left|\frac{z}{s}\right|^n \leq \frac{1}{R_0 - |z|}\left(\frac{|z|}{R_0}\right)^n$$

$s \in C_0$
$\Rightarrow |s-z| \geq R_0 - |z|$

f is analytic on C_0

$\Rightarrow f$ is continuous on C_0

$\Rightarrow \exists M > 0$ s.t. $|f(s)| \leq M \quad \forall s \in C_0$

In general, $z_0 \neq 0$, then we let $g(z) = f(z+z_0)$,

then apply the pervious result to $g(z)$, the result follows.

Conclusion: f is analytic throughout an open disk $\{z\in\mathbb{C} : |z-z_0|<R\}$.

$\Rightarrow$ f has a power series representation throughout that open disk.

Natural question: Uniqueness ?

Theorem:

If there exist constants a_n, $n=0,1,2,\cdots$ such that $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for all points $z\in\{z\in\mathbb{C} : |z-z_0|<R\}$, then the power series must be the Talyor series, i.e. $a_n=\frac{f^{(n)}(z_0)}{n!}$.

(prove later !)

e.g. $f(z)=e^z$ is an entire function

$f^{(n)}(z)=e^z$ and $f^{(n)}(0)=1$ for $n=0,1,2,\cdots$

$e^z=\sum_{n=1}^{\infty}\frac{z^n}{n!}$ $\quad\forall z\in\mathbb{C}$

In particular, $z=x+0i$, $\quad e^x=\sum_{n=1}^{\infty}\frac{x^n}{n!}$ $\quad\forall x\in\mathbb{R}$.

Ex:

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \qquad \forall z\in\mathbb{C}$$

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} \qquad \forall z\in\mathbb{C}$$

$$\frac{1}{1-z} = 1+z+z^2+z^3+\cdots=\sum_{n=0}^{\infty}z^n \qquad \forall |z|<1$$

e.g. $\sin 2z = 2z-\frac{(2z)^3}{3!}+\frac{(2z)^5}{5!}-\frac{(2z)^7}{7!}+\cdots = 2z-\frac{4}{3}z^3+\frac{4}{15}z^5-\frac{8}{315}z^7+\cdots$ $\qquad \forall z\in\mathbb{C}$

e.g.

$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots=$

$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$

$$2\sin z\cos z \overset{?}{=} 2\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)$$

$$=2z+2\left(-\frac{1}{2!}-\frac{1}{3!}\right)z^3+2\left(\frac{1}{4!}+\frac{1}{3!2!}+\frac{1}{5!}\right)z^5+\cdots$$

$$=2z-\frac{4}{3}z^3+\frac{4}{15}z^5-\frac{8}{315}z^7+\cdots$$

III) Power Series

Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

Things to be studied :

1) Region of convergence ?

2) Absolute / Uniform convergent ?

3) If $\sum_{n=0}^{\infty} a_n(z-z_0)^n = f(z)$, what are the properties of f ? (continuous ? analytic ?)

Theorem :

If a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges when $z=z_1$ $(z_1 \neq z_0)$, then it is absolutely convergent at each point $z \in \{z \in \mathbb{C} : |z-z_0| < R_1\}$, where $R_1 = |z_1 - z_0|$.

i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} |a_k(z-z_0)^k|$ exists .

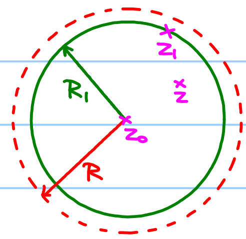
proof :

$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges

$\Rightarrow$ $|a_n(z_1-z_0)^n|$ is bounded for all $n = 0, 1, 2, \cdots$

i.e. $\exists M>0$ s.t. $|a_n(z_1-z_0)^n| \leq M$

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n \leq M\rho^n \qquad \text{where } \rho = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$$

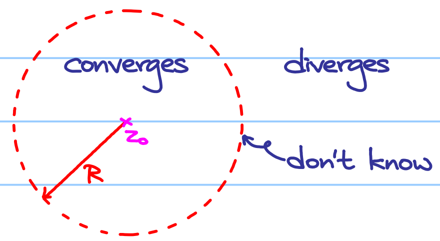


By comparsion test, $\lim_{n\to\infty} \sum_{k=1}^{n} |a_k(z-z_0)^k|$ exists.

Direct consequence :

Note : Absolutely convergent $\Rightarrow$ convergent

For a power series , we have



$R = \sup\{ r : \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z with $|z-z_0| < r\}$ is said to be the radius of convergence of the power series .

$|z-z_0| = R$ is said to be the circle of convergence of the power series .

Furthermore , if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z \in \mathbb{C}$, then we say $R = +\infty$;

if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges only when $z = z_0$, then we say $R = 0$.

Theorem :

If z_1 is a point inside the circle of convergence $|z-z_0|=R$ of a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ then it is uniformly convergent in $\{z\in\mathbb{C} : |z-z_0|\le R_1\}$, where $R_1 = |z_1-z_0|$.

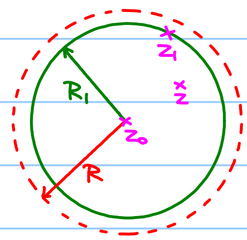
proof :

By assumption, for every $z\in\{z\in\mathbb{C} : |z-z_0|<R\}$, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ exists. Let $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$

Rewrite :

$\forall z\in\{z\in\mathbb{C} : |z-z_0|\le R_1\}$, $\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$ s.t. $\left|\sum_{k=1}^{n} a_k(z-z_0)^k - f(z)\right|<\varepsilon \quad \forall n\ge N(\varepsilon)$.

By assumption, $\exists z'$ s.t. $|z_1-z_0|<|z'-z_0|<R$



Previous theorem $\Rightarrow \sum_{n=0}^{\infty}|a_n(z_1-z_0)^n|$ converges (to a real number L)

Let $\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$ s.t. $\underbrace{\left|\sum_{k=1}^{n}|a_k(z_1-z_0)^k| - L\right|}_{=\ \sum_{k=n+1}^{\infty}|a_k(z_1-z_0)^k|} < \varepsilon \quad \forall n\ge N(\varepsilon)$

Let $z\in\{z\in\mathbb{C} : |z-z_0|\le R_1\}$,

Fix n,

$$|z-z_0|\le|z_1-z_0|$$

$$|z-z_0|^k\le|z_1-z_0|^k \quad \forall k\in\mathbb{N}$$

$$|a_k(z-z_0)^k|\le|a_k(z_1-z_0)^k| \quad \forall k\in\mathbb{N}$$

$$\sum_{k=n+1}^{m}|a_k(z-z_0)^k| \le \sum_{k=n+1}^{m}|a_k(z_1-z_0)^k|$$

By comparsion test,

$$\sum_{k=n+1}^{\infty}|a_k(z-z_0)^k| \le \sum_{k=n+1}^{\infty}|a_k(z_1-z_0)^k|$$

Also, $\left|\sum_{k=n+1}^{m} a_k(z-z_0)^k\right| \le \sum_{k=n+1}^{m}|a_k(z-z_0)^k|$

$$\underbrace{\left|\sum_{k=n+1}^{\infty} a_k(z-z_0)^k\right|}_{=\ \left|f(z)-\sum_{k=1}^{n} a_k(z-z_0)^k\right|} = \lim_{m\to\infty}\left|\sum_{k=n+1}^{m} a_k(z-z_0)^k\right| \le \lim_{m\to\infty}\sum_{k=n+1}^{m}|a_k(z-z_0)^k| = \sum_{k=n+1}^{\infty}|a_k(z-z_0)^k|$$

$$\therefore \quad \left|f(z)-\sum_{k=1}^{n} a_k(z-z_0)^k\right| \le \sum_{k=n+1}^{\infty}|a_k(z-z_0)^k| \le \sum_{k=n+1}^{\infty}|a_k(z_1-z_0)^k| < \varepsilon \quad \forall n\ge N(\varepsilon)$$

Corollary:

$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a continuous function in $\{z\in\mathbb{C} : |z-z_0|<R\}$,

where R is the radius of convergence.

proof:

Let $z_1\in\{z\in\mathbb{C} : |z-z_0|<R\}$, $\varepsilon>0$,

Let $f_n(z)=\sum_{k=0}^{n} a_k(z-z_0)^k$

- $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniformly converges $\Rightarrow \exists N(\varepsilon)\in\mathbb{N}$ s.t. $|f(z)-f_n(z)|$, $|f_n(z_1)-f(z_1)|\leq\frac{\varepsilon}{3}$ $\forall n\geq N(\varepsilon)$

  pick $K\geq N(\varepsilon)$.

- $f_K(z)$ is a polynomial, which is continuous, $\exists\,\delta>0$ s.t. $|f_K(z)-f_K(z_1)|<\frac{\varepsilon}{3}$ $\forall\ |z-z_1|<\delta$

$$
\begin{aligned}
|f(z)-f(z_1)| &= |f(z)-f_K(z)+f_K(z)-f_K(z_1)+f_K(z_1)-f(z_1)|\\
&\leq |f(z)-f_K(z)|+|f_K(z)-f_K(z_1)|+|f_K(z_1)-f(z_1)|\\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
\end{aligned}
$$

Theorem:

Let C be any contour interior to the circle of convergence of the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, and let $g(z)$ be any continuous on C. Then

$$\int_C g(z)f(z)\,dz=\sum_{k=0}^{\infty} a_k\int_C g(z)(z-z_0)^k\,dz$$

proof:

Note: $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ is uniformly convergent on C.

Let $\varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$ such that $\left|f(z)-\sum_{k=0}^{n-1} a_k(z-z_0)^k\right|<\varepsilon$ $\forall\ n\geq N(\varepsilon)$, $z\in C$

$$\left|\int_C f(z)g(z)\,dz-\sum_{k=0}^{n-1} a_k\int_C g(z)(z-z_0)^k\,dz\right|$$

$=\left|\int_C g(z)\left(f(z)-\sum_{k=0}^{n-1} a_k(z-z_0)^k\right)dz\right|$ $\quad$ ($\exists M>0$ s.t. $|g(z)|<M$ $\forall z\in C$)

$\leq ML\varepsilon$ $\quad$ $\forall\ n\geq N(\varepsilon)$

Corollary:

$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is an analytic function in $\{z\in\mathbb{C} : |z-z_0|<R\}$,

where R is the radius of convergence.

proof:

Take $g(z)\equiv 1$

$$\int_C f(z)\,dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k dz = 0$$

$\Rightarrow f(z)$ is analytic in $\{z\in\mathbb{C} : |z-z_0|<R\}$.

Theorem:

If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R\}$, where R is the radius of convergence,

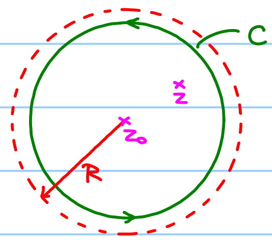
then $f'(z)=\sum_{n=1}^{\infty} n a_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R\}$.

proof:

Let $z\in\{z\in\mathbb{C} : |z-z_0|<R\}$.

Choose C such that it lies in $\{z\in\mathbb{C} : |z-z_0|<R\}$

and z lies in the interior of C.



Take $g(s)=\frac{1}{2\pi i}\cdot\frac{1}{(s-z)^2}$

$$\int_C g(s)f(s)\,ds = \sum_{k=0}^{\infty} a_k \int_C g(s)(s-z_0)^k ds$$

$$\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds = \sum_{k=0}^{\infty} a_k\left(\frac{1}{2\pi i}\int \frac{(s-z_0)^k}{(s-z)^2}ds\right)$$

$$f'(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty} k a_k (z-z_0)^{k-1}$$

Recall: Cauchy Integral formula

$$\frac{1}{2\pi i}\int_C \frac{h(s)}{(s-z)^2}ds = h'(z)$$

Consider $h(s)=f(s)$ on LHS

$h(s)=(s-z_0)^n$ on RHS.

e.g. Verification

$$\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}$$

$$\frac{d}{dz}\sin z = \cos z$$

$$\frac{d}{dz}\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!} = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots$$

Theorem: (Uniqueness of series representation)

If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R\}$, where R is the radius of convergence,

then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the Talyor series expansion for f at z_0, i.e. $a_n=\frac{f^{(n)}(z_0)}{n!}$ $\forall n\geq 0$

proof: Take $g(z)=\frac{1}{2\pi i}\frac{1}{(z-z_0)^{n+1}}$ for $n\geq 0$.

$$\int_C g(z)f(z)\,dz=\sum_{k=0}^{\infty} a_k\int_C g(z)(z-z_0)^k\,dz$$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz=\sum_{k=0}^{\infty} a_k\frac{1}{2\pi i}\int_C (z-z_0)^{k-n-1}\,dz$$

Note: $\frac{1}{2\pi i}\int_C (z-z_0)^{k-n-1}\,dz=\begin{cases}1 & \text{if } k=n\\ 0 & \text{otherwise}\end{cases}$

$$\frac{f^{(n)}(z_0)}{n!}=a_n$$

Theorem: (Multiplication)

If $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R_f\}$

$g(z)=\sum_{n=0}^{\infty} b_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R_g\}$

where R_f and R_g are the radii of convergence of f and g respectively.

then $f(z)g(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R\}$, where $R=\min\{R_f,R_g\}$

and $c_n=\sum_{k=0}^{n} a_k b_{n-k}$ $\forall n\geq 0$

proof:

Note: f and g are analytic in $\{z\in\mathbb{C} : |z-z_0|<R\}$,

$\therefore$ $h(z)=f(z)g(z)$ has a series expansion $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ in $\{z\in\mathbb{C} : |z-z_0|<R\}$ and $c_n=\frac{h^{(n)}(z_0)}{n!}$

$$h^{(n)}(z)=\sum_{k=0}^{n} C_k^n f^{(k)}(z)\,g^{(n-k)}(z)$$

$$=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(k)}(z)\,g^{(n-k)}(z)$$

$$\therefore\ \frac{h^{(n)}(z_0)}{n!}=\sum_{k=0}^{n}\frac{f^{(k)}(z_0)}{k!}\frac{f^{(n-k)}(z_0)}{(n-k)!}$$

$$c_n=\sum_{k=0}^{n} a_k b_{n-k}$$

e.g. $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \quad \forall z\in\mathbb{C}$

$\frac{1}{1+z} = 1-z+z^2-z^3+\cdots \quad \forall |z|<1$

$$\frac{e^z}{1+z} = \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1-z+z^2-z^3+\cdots\right)$$

$$\begin{aligned} &= 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \\ &\quad -z-z^2-\frac{z^3}{2!}-\frac{z^4}{3!}-\cdots \\ &\quad +z^2+z^3+\frac{z^4}{2!}+\cdots \\ &\quad -z^3-z^4-\cdots \\ &= 1+\frac{1}{2}z^2-\frac{1}{3}z^3+\cdots \end{aligned}$$

e.g. Prove if f is analytic at z_0 and $f(z_0)=f'(z_0)=\cdots=f^{(m)}(z_0)=0$, then the function

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & \text{when } z\neq z_0 \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z=z_0 \end{cases}$$ is analytic at z_0.

Note: f is analytic at z_0

$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ in an open disk centered at z_0.

$$= \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= \sum_{l=0}^{\infty} \frac{f^{(l+m+1)}(z_0)}{(l+m+1)!}(z-z_0)^{l+m+1}$$

$$\frac{f(z)}{(z-z_0)^{m+1}} = \sum_{l=0}^{\infty} \frac{f^{(l+m+1)}(z_0)}{(l+m+1)!}(z-z_0)^{l} \quad \text{if } z\neq z_0$$

$\Rightarrow \sum_{l=0}^{\infty} \frac{f^{(l+m+1)}(z_0)}{(l+m+1)!}(z-z_0)^{l}$ is a convergent power series in that open disk

and $\sum_{l=0}^{\infty} \frac{f^{(l+m+1)}(z_0)}{(l+m+1)!}(z-z_0)^{l}$ converges to $g(z)$

$\Rightarrow g(z)$ is analytic at z_0

Note: It implies if f is analytic at z_0 and $f(z_0)=f'(z_0)=\cdots=f^{(m)}(z_0)=0$,
then $f(z)=(z-z_0)^m g(z)$ for some function $g(z)$ which is analytic at z_0.